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HARVARD UNIVERSITY

SEEPAGE THROUGH DAMS

BY

ARTHUR CASAGRANDE

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SEEPAGE THROUGH DAMS.

BY ARTHUR CASAGRANDE.*

[Read February 28, 1935.]

CONTENTS.

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A. Introduction.

Until about ten years ago, the design of earth dams and dikes was based almost exclusively on empirical knowledge and consisted largely of adopting the cross-section of successful dams, with little regard to differences in character of soil and foundation conditions. At present, we are in a period in which the behavior of dams, particularly those which have failed, is analysed in the light of modern soil mechanics. The understanding and knowledge thus accumulating is being used as the basis for a more scientific approach to the design of such earth structures.

The most outstanding progress in this subject relates to the question of seepage beneath dams and dikes and to the effect of seepage on the stability of these structures. Foundation failures due to seepage, commonly known as "piping," were, for the first time, correctly explained by Terzaghi (1)* who developed what may be termed the "mechanics of piping." Later, Terzaghi (2, 3 and 4), called attention to the importance of the forces created within earth dams and concrete dams, due to the percolation of water. The practical application of this information has lagged behind our understanding of these forces partly because of theoretical difficulties of analyzing problems of seepage. It is only in recent years that substantial progress has been made in the solution of problems of seepage and ground water flow with free or open surface, of the flow through anisotropic materials, and of the conditions of flow through joint planes of different materials.

B. Darcy's Law for the Flow of Water through Soils.

The flow of water through soils, so far as it affects the question of seepage through dams, follows Darcy's empirical law, which states that *the amount of flow is directly proportional to the hydraulic gradient.* This law can be expressed either in the form:

$$v_o = k\,i \quad \text{or} \quad Q = k\,i\,A\,t \qquad (1)$$

in which the symbols have the following meaning:

v_o = discharge velocity†
k = coefficient of permeability
i = hydraulic gradient
Q = quantity of water
A = area
t = time

In Figure 1 the meaning of Darcy's law is illustrated in simple form. A prismatic or cylindrical soil sample is exposed on the left side to a head

*Numerals refer to the bibliography at the end of this paper.

†This must not be confused with the seepage velocity $v_s = v_o \frac{1+e}{e}$, in which e = ratio of volume of voids to volume of solid matter. The average velocity through the soil is represented by the seepage velocity, while the discharge velocity determines the quantity of flow.

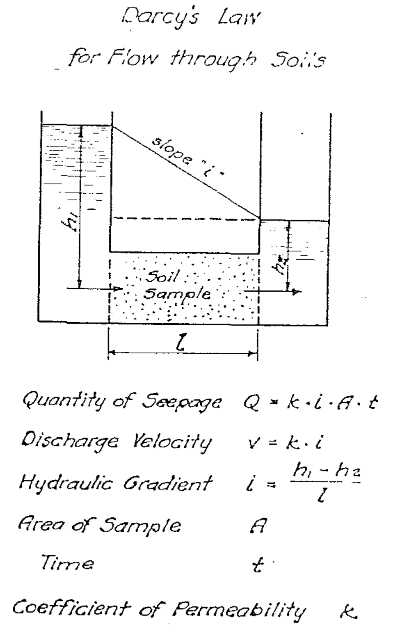

Fig. 1.—Darcy's Law for Flow through Soils.

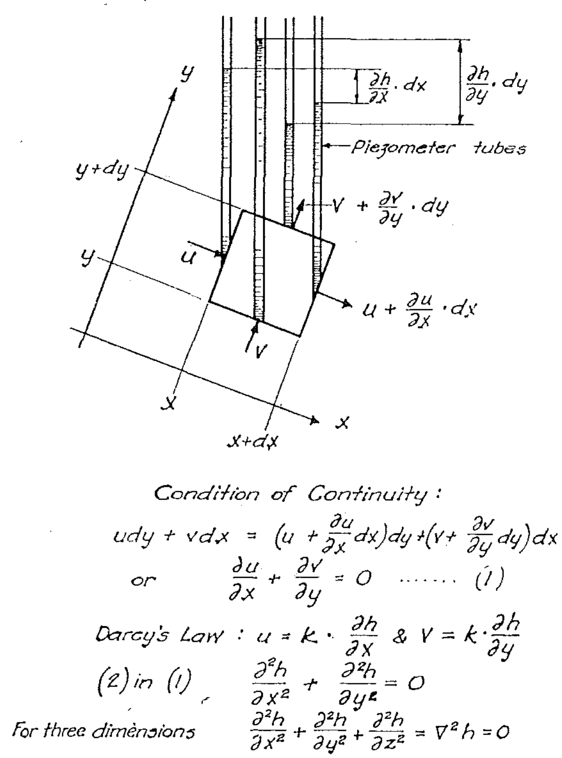

Fig. 2.—General Differential Equation for the Flow of Water through Homogeneous Isotropic Soils.

of water h_1, and on the right side to a smaller head h_2. As a result, water will flow through the sample at a rate directly proportional to the hydraulic gradient, $i = \frac{h_1 - h_2}{l}$. If, for example, the difference in head $(h_1 - h_2)$ in Figure 1 is doubled, the quantity of seepage will also be doubled. This linear relationship suggests that the flow of water through the voids of most soils possesses the characteristics of laminar flow.

Darcy's law is frequently attacked as being incorrect. In general these attacks are based on misinterpretation of test results or improper technique of testing. In many cases they are due to loss of internal stability of the soil under the action of flowing water.

The reader may be assured that this law is valid for the study of seepage through dams.

C. General Differential Equation for the Flow of Water through Homogeneous Soil.

If water is percolating through a homogeneous mass of soil in such a manner that the voids of the soil are completely filled with water and no change in the size of the voids takes place, the quantity entering from one or several directions into a small element of volume of the soil (as shown in Figure 2) must be equal to the amount of water flowing out on the other faces of this element of volume during any given element of time. This condition, which is a statement of the fact that both water and soil are incompressible, can be expressed for the three-dimensional case by the following equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2)$$

This is known as the *equation of continuity*. (See Reference 5.) In this equation, u, v and w are the three components of the discharge velocity v_o. If $\frac{dh}{dl}$ represents the hydraulic gradient in the direction of flow and $\frac{dh}{dx}$, $\frac{dh}{dy}$ and $\frac{dh}{dz}$ are its three components, then Darcy's law:

$$v_o = k\frac{dh}{dl}$$

can also be expressed by the following equations:

$$\left.\begin{aligned} u &= k\frac{\partial h}{\partial x} \\ v &= k\frac{\partial h}{\partial y} \\ w &= k\frac{\partial h}{\partial z} \end{aligned}\right\} \qquad (3)$$

By substituting Equation (3) in Equation (2), one arrives at the general differential equation for the steady flow of water through isotropic soils. This has the form of a Laplace differential equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \qquad (4)$$

In our problem of seepage through dams we have to deal only with the two-dimensional case which is satisfied by the equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (4a)$$

This equation represents *two families of curves intersecting at right angles.* (See Reference 5, p. 24 and 25.) In hydro-mechanics these curves are known respectively as the *flow lines* and the *equipotential lines* (or lines of equal head).

D. Forchheimer's Graphical Solution.

Although the general differential equation (4) has been solved only for few and simple cases of seepage, we can make use of certain geometric

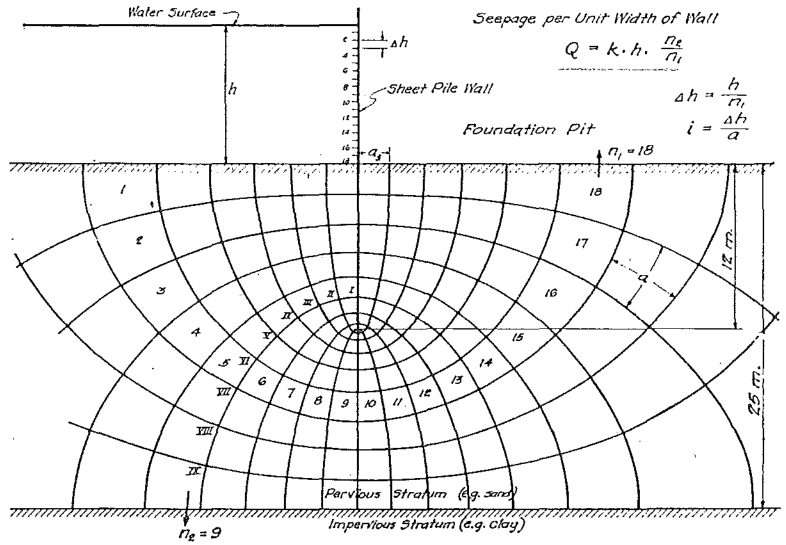

FIG. 3.—FLOW NET BENEATH SHEET PILE WALL.

properties of flow lines and equipotential lines that permit graphical solutions for practically all two-dimensional seepage problems. This method was devised by Forchheimer (5), twenty-five years ago.

To explain this graphical method, the problem of determining the seepage beneath a sheet pile wall, shown in Figure 3 is chosen. The ground surface is a line of equal head or an equipotential line; the head being equal to the height of water standing above the ground surface, which is h on the left side, and zero on the right side of the wall. The bottom of the pervious soil stratum is a flow line; incidentally the longest flow line. The sides of the sheet pile wall and the short width at the bottom of the wall are the shortest flow line.

If, from the infinite number of flow lines possible within the given area, we choose only a few in such a manner that the same fraction Δq of the total seepage is passing between any pair of neighboring flow lines, and, similarly, if we choose from the infinite number of possible equipotential lines only a few in such a manner that the drop in head Δh between any pair of neighboring equipotential lines is equal to a constant fraction of the total loss in head h, then the resulting "flow net," Figure 3, possesses the property that the ratio of the sides of each rectangle, bordered by two flow lines and two equipotential lines, is constant. (See Reference 5, p. 82.) If all sides of one such rectangle are equal, then the entire flow net must consist of squares. Conversely, it can be proved that if one succeeds in plotting two sets of curves so that they intersect at right angles, forming squares and fulfilling the boundary conditions, then one has solved, graphically, equation (4a) for this problem. With experience, this method can be applied successfully to the most complicated problems of seepage and ground water flow in two dimensions, including seepage with a free surface.

After having plotted a flow net that fulfills satisfactorily these necessary conditions, one can derive therefrom, by simple computations, any desired information on quantity of seepage, seepage pressures, and hydrostatic uplift. For example, the total seepage per unit of length and per unit of time is determined from the following formula, which is simple to derive from Darcy's law:

$$q = kh\frac{n_2}{n_1} \qquad (5)$$

in which n_1 is the number of squares between two neighboring flow lines, and n_2 the number of squares between two neighboring equipotential lines.

The maximum hydraulic gradient on the discharge surface, which influences the safety against "piping" or "blows," is equal to:

$$i_s = \frac{\Delta h}{a_s} \qquad (6)$$

in which a_s is the length of the smallest square on the discharge surface, as indicated in Figure 3, and $\Delta h = \frac{h}{n_1}$, the drop in head between two adjacent equipotential lines.

To assist the beginner in learning the graphical method, the following suggestions are made:

1. Use every opportunity to study the appearance of well-constructed flow nets; when the picture is sufficiently absorbed in your mind, try to draw the same flow net without looking at the available solution; repeat this until you are able to sketch this flow net in a satisfactory manner.

2. Four or five flow channels are usually sufficient for the first attempts; the use of too many flow channels may distract the attention from the essential features. (For examples see Figures 18b and c.)

3. Always watch the appearance of the entire flow net. Do not try to adjust details before the entire flow net is approximately correct.

4. Frequently there are portions of a flow net in which the flow lines should be approximately straight and parallel lines. The flow channels are then about of equal width, and the squares are therefore uniform in size. By starting to plot the flow net in such an area, assuming it to consist of straight lines, one can facilitate the solution.

5. The flow net in confined areas, limited by parallel boundaries, is frequently symmetrical, consisting of curves of elliptical shape. (For example see Figure 3.)

6. The beginner usually makes the mistake of drawing too sharp transitions between straight and curved sections of flow lines or equipotential lines. Keep in mind that all transitions are smooth, of elliptical or parabolic sharpe. The size of the squares in each channel will change gradually.

7. In general, the first assumption of flow channels will not result in a flow net consisting throughout of squares. The drop in head between neighbouring equipotential lines corresponding to the arbitrary number of flow channels, will usually not be an integer of the total drop in head. Thus, where the flow net is ended, a row of rectangles will remain. For usual purposes this has no disadvantages, and the last row is taken into consideration in computations by estimating the ratio of the sides of the rectangles. If, for the sake of appearance, it is desired to resolve the entire area into squares, then it becomes necessary to change the number of flow channels, either by interpolation or by a new start. One should not attempt to force the change into squares by adjustments in the neighbouring areas, unless the necessary correction is very small.

8. Boundary conditions may introduce singularities into the flow net, which are discussed more in detail in Appendix I, c.

9. A discharge face, in contact with air, is neither a flow line nor an equipotential line. Therefore, the squares along such a boundary are incomplete. However, such a boundary must fulfill the same condition as the line of seepage regarding equal drops in head between the points where the equipotential lines intersect.

10. When constructing a flow net containing a free surface one should start by assuming the discharge face and the discharge point and then work toward the upstream face until the correct relative positions of entrance point and discharge toe are attained. Hence, the scale to which a flow net with a free surface is plotted, will not be known until a large portion of the flow net is finished. For seepage problems with a free surface it is practically impossible to construct a flow net to a predetermined scale in a reasonable length of time.

E. Seepage through Dams; General Considerations.

In almost all problems concerning seepage *beneath* sheet pile walls or *through the foundation* of a dam; *all boundary conditions are known*. However, in the *seepage through* an earth dam or dike, the *upper boundary* or uppermost flow line is *not known*, but must first be found, thus introducing a complication. This upper boundary is a free water surface and will be referred to as the *line of seepage*.

Among the available theoretical solutions for seepage with a free surface there is one case which is of particular importance in connection with our problem. It is Kozeny's solution (6) of the flow along a horizontal impervious stratum that continues at a given point into a horizontal pervious stratum, thus representing an open horizontal discharge surface as shown in Figure 9d. In this case, all flow lines, including the line of seepage, and all equipotential lines are confocal parabolas with point A as the focus.

For the more common problem of seepage through cross-sections in which the discharge slope forms an angle with the horizontal between 0° and 180°, such as the open discharge on the downstream face of a dam or discharge into an overhanging slope of a very pervious toe, such as a rock

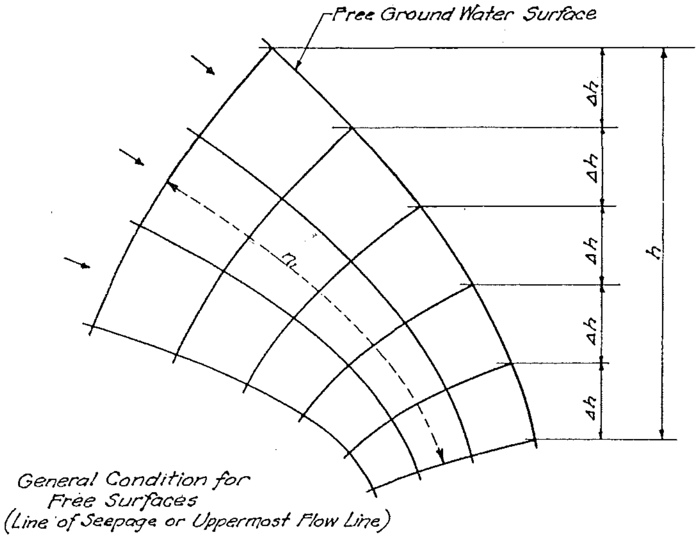

FIG. 4.— GENERAL CONDITION FOR LINE OF SEEPAGE.

fill toe, one has to use either a graphical solution based on the construction of the flow net, or some approximate mathematical solution. In either case one must introduce certain conditions that the free water surface or line, of seepage must always fulfill.

The first condition is that the elevation of the point of intersection of any equipotential line with the line of seepage represents the head along this equipotential line. If we construct a flow net consisting of squares, then it follows that all intersections of equipotential lines with the line of seepage must be equidistant in the vertical direction. These distances, illustrated in Figure 4, represent the actual drop in head $\Delta h = \frac{h}{n_1}$ between any two neighboring equipotential lines.

The second condition refers to the slope of the line of seepage at the point of intersection with any boundary, as for example at the points of entrance and discharge and at the boundary line between different soils

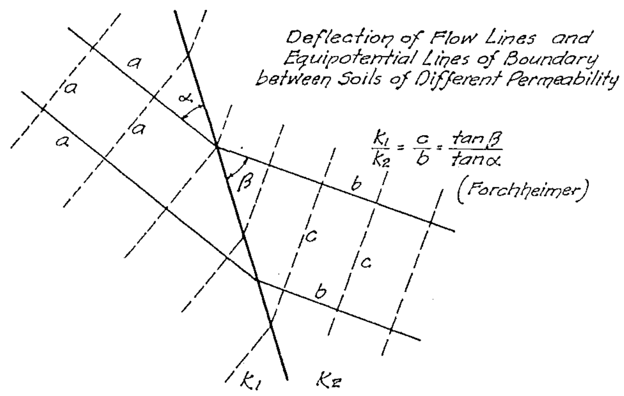

FIG. 5.— DEFLECTION OF FLOW NET AT BOUNDARY OF SOILS OF DIFFERENT PERMEABILITY.

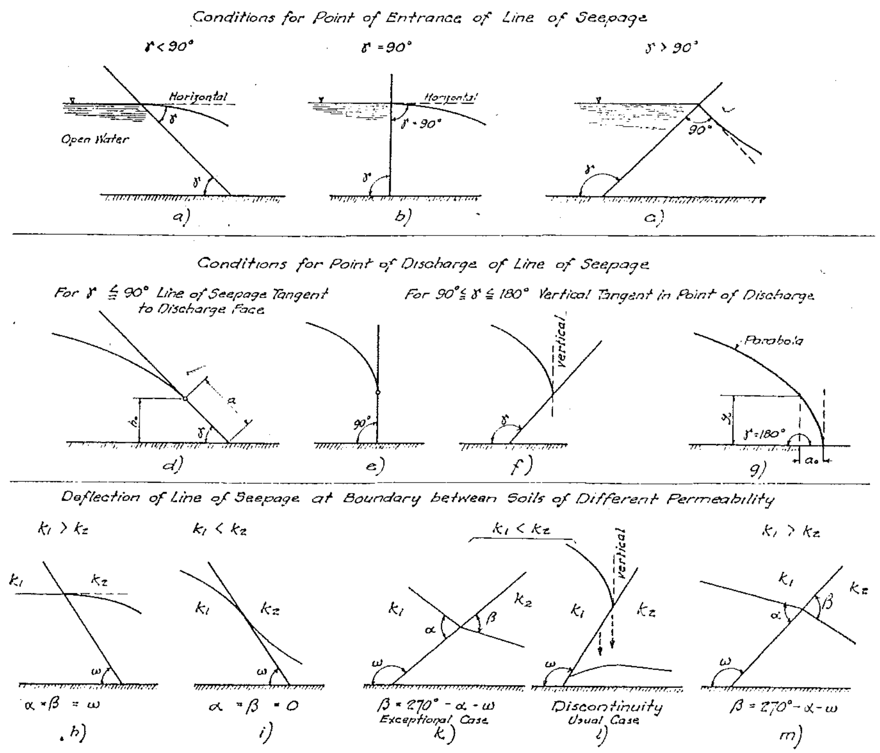

FIG. 6.— ENTRANCE, DISCHARGE AND TRANSFER CONDITIONS OF LINE OF SEEPAGE.

(See Figure 5). By considerations based on the general properties of a flow net, one can arrive at the conditions which must be fulfilled at such points of transfer. In Appendix I are assembled derivations for typical cases. If, for example, the downstream face is inclined less than or equal to 90°, one finds that the line of seepage must be tangent to that face at the discharge point. However, for all overhanging slopes, the tangent at the discharge point must be vertical. A summary of the possible combinations is assembled in Figure 6.

F. Seepage through Homogeneous Isotropic Earth Dams.

a. Approximate Solution for $\alpha < 30°$. The first approximate mathematical solution for determining the quantity of seepage and the line of seepage through a homogeneous earth section on an impervious base was

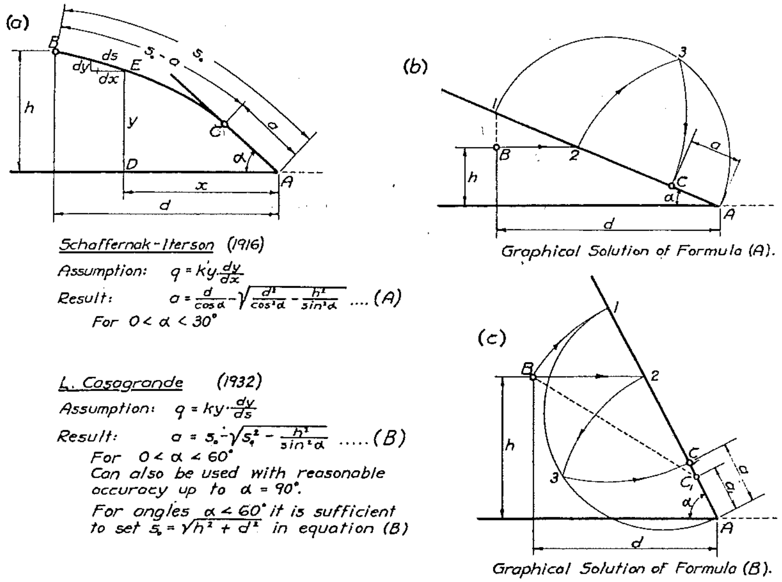

FIG. 7.— GRAPHICAL DETERMINATION OF DISCHARGE POINT FOR $\alpha < 60°$.

proposed independently in 1916 by Schaffernak (7) in Austria, and Iterson (8), in Holland. It is based on Dupuit's (9) assumption* that in every point of a vertical line the hydraulic gradient is constant and equal to the slope $\frac{dy}{dx}$ of the line of seepage at its intersection with that vertical line. This assumption represents a good approximation for the average hydraulic gradient in such a vertical line providing the slope of the line of seepage is relatively flat.

*On this same assumption are based the common methods of computing ground water flow toward wells.

With this assumption and the condition that the quantity of water flowing through any cross-section per unit of time must be constant, one can derive the differential equation for the line of seepage from Figure 7a:

$$q = ky\frac{dy}{dx} \qquad (7)$$

The solution of equation (7) yields the equation of a parabola. Assuming that the quantities h, d, and α, in Figure 7, are known, and with the boundary conditions $y=h$ for $x=d$, and $dy/dx=\tan\alpha$ for $x=a\cos\alpha$, or $y=a\sin\alpha$, integration leads to the following formula for the distance a which determines the discharge point C of the line of seepage on the downstream face of the dam:

$$a = \frac{d}{\cos\alpha} - \sqrt{\frac{d^2}{\cos^2\alpha} - \frac{h^2}{\sin^2\alpha}} \qquad (8)$$

$$q = k\, a \sin\alpha \tan\alpha \qquad (9)$$

These equations differ from their original form in the use of the distance a, instead of its vertical projection, a change which provides a common basis for all theoretical developments in this paper. A further advantage

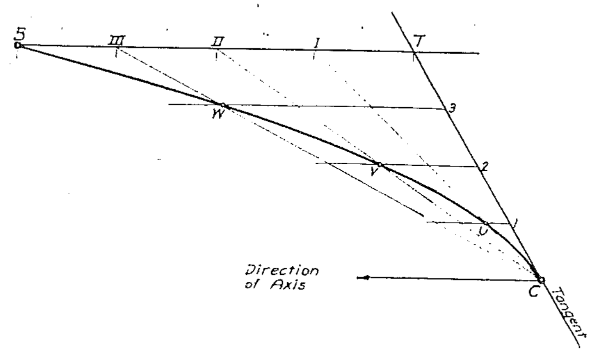

FIG. 8.— METHOD OF LOCATING POINTS ON A PARABOLA.

of this change resides in the possibility of determining graphically the distance a by means of a simple construction* which is shown in Figure 7b. The ordinate through the known point B of the line of seepage is extended to its intersection *1* with the discharge slope, and a semi-circle drawn through the points *1* and A, with its center on the discharge slope. Then a horizontal line through B is intersected with the discharge slope in point *2*, and the distance *2-A* projected onto the circle, yielding point *3*. The final step is to project the distance *1-3* onto the discharge slope. This yields the desired discharge point C. The proof for the validity of this method is readily found by comparing this construction with equation (8) and need not be discussed in detail.

*This construction is a simplification of another method which was proposed in References 10 and 11.

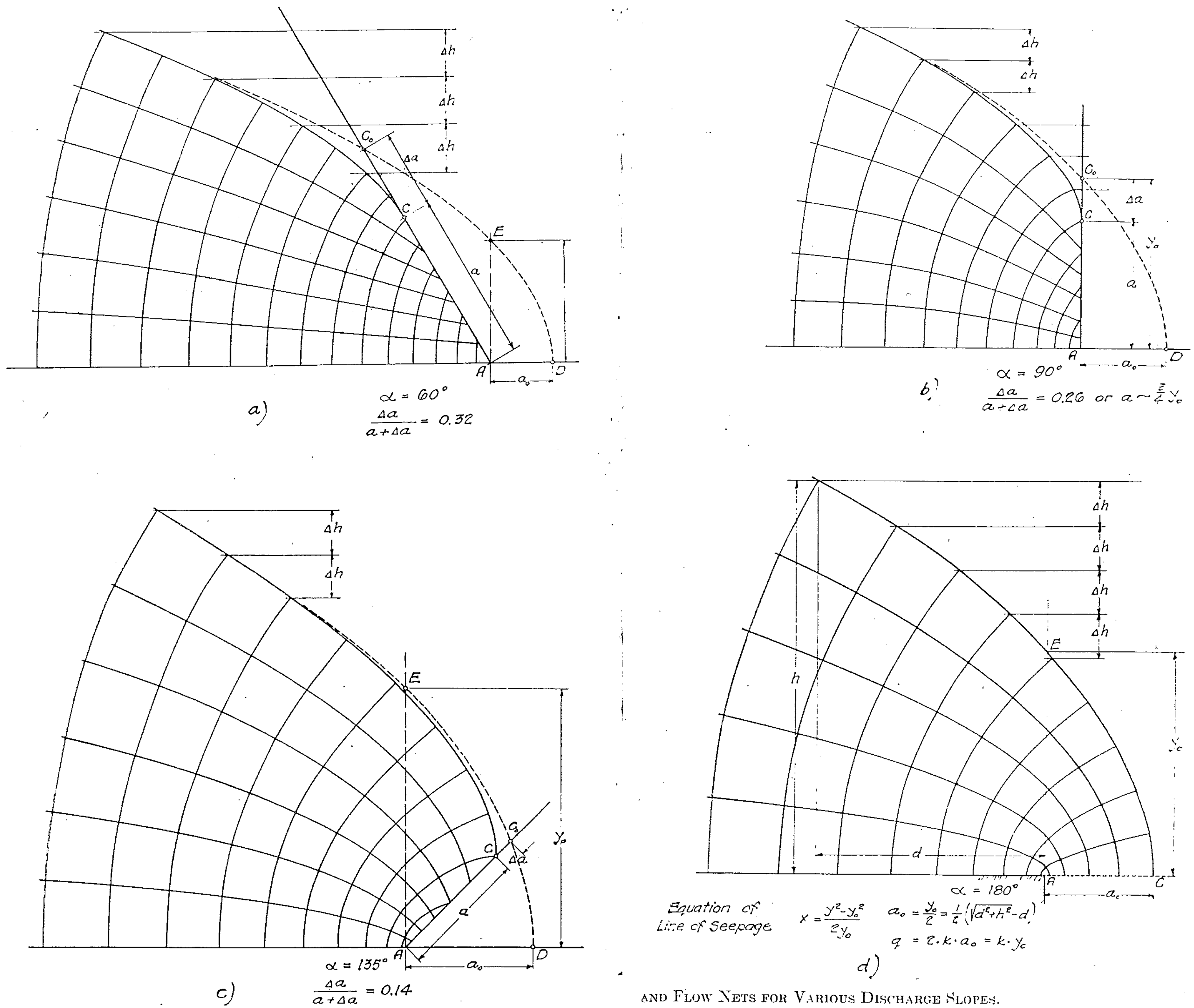

FIG. 9.—COMPARISON BETWEEN BASIC PARABOLA AND FLOW NETS FOR VARIOUS DISCHARGE SLOPES.

For the solution of many problems it is sufficient to know the discharge point of the line of seepage. If it is desired to draw the entire line of seepage, from the known point B to the discharge point C, one can make use of the graphical method, shown in Figure 8, for the rapid construction of any number of points on a parabola for which are known two points, the tangent to the parabola at one of these points, and the direction of the axis.

Through point B, Figure 8, one draws a line parallel to the axis and determines its intersection T with the tangent. Then one divides the distances B-T and C-T into an arbitrary number of equal parts, such as four parts. Points I, II and III are then connected with point C, and through points 1, 2 and 3 one draws lines parallel to the axis. The points where the lines through 1 and I, 2 and II, etc., intersect, are points of the parabola.

b. Approximate Solution for $\alpha > 30°$. The approximate solution by means of equation (8), or the corresponding graphical method shown in Figure 7b, gives satisfactory results for slopes of $\alpha < 30°$. For steeper slopes the deviation from the correct values increases rapidly beyond tolerable limits.

The causes for this deviation become apparent from a study of the flow net for a slope of $\alpha = 60°$, shown in Figure 9a. One can see that in the vicinity of the discharge point the size of the squares along the vertical line through the discharge point decreases only slightly towards the base. The average hydraulic gradient along this vertical line is larger than the hydraulic gradient $\frac{dy}{ds}$ along the line of seepage by less than 10 per cent. However, the sine of 60°, which is the true hydraulic gradient for the line of seepage at the discharge point, is only about one-half of the tangent of 60° used according to Dupuit's assumption. Hence the seepage can be analysed with a satisfactory degree of accuracy by means of the following equation:

$$q \doteq ky\frac{dy}{ds} \qquad (10)$$

This improvement was proposed by Leo Casagrande (10).

The difference between the use of the tangent and the sine of the slope of the line of seepage is best illustrated by the following numerical comparison for various angles:

Slope	tan	sin
30°	0.577	0.500
60°	1.732	0.866
90°	∞	1.000

Hence, for slopes <30° both methods may be used for practical purposes with equal advantage. For slopes > 30° the deviation by using $\frac{dy}{dx}$ becomes intolerably large, while the use of $\frac{dy}{ds}$ is very satisfactory for slopes up to 60°; and if deviations of 25 per cent. are permitted, it may even be used up to 90°, that is, for a vertical discharge face.

Gilboy (12) succeeded in finding an implicit solution of equation (10) which is recommended where greater accuracy is required than can be obtained by means of the graphical solution. The errors involved in the position of the discharge point, as obtained by one or the other method from equation (10), were investigated by G. P. Reyntjiens (13).

Using the symbols shown in Figure 7a, and assuming that in each vertical the hydraulic gradient is equal to $\frac{dy}{ds}$, equation (10) is the differential equation for the line of seepage. The solution of this equation cannot readily be expressed by rectangular coördinates x and y. (See References 12 and 13.) However, the use of s and s_o, measured along the line of seepage, does not represent any practical difficulty in the actual application of this method. The quantity a, which determines the discharge point for the line of seepage, is found by a simple integration:

$$qs = \frac{ky^2}{2} + \text{constant}$$

$$\text{Boundary Conditions} \begin{cases} s = a, \; y = a \sin\alpha, \; q = ka \sin^2\alpha \\ s = s_o, \; y = h \end{cases}$$

$$a = s_o - \sqrt{s_o^2 - \frac{h^2}{\sin^2\alpha}} \qquad (11)$$

$$q = k\, a \sin^2\alpha \qquad (12)$$

Again, the quantities employed in these equations differ from the original form as presented in References 10 and 11 to permit a simple graphical solution. This graphical solution of equation (11) is illustrated in Figure 7c, and can be easily verified. It requires first an assumption for the discharge point. The length $(s_o - a)$ is simply taken equal to the straight line from B to C_1, shown as a dotted line in Figure 7c. The slight error which is introduced when $(s_o - a)$ is replaced by a straight line has a negligible effect on the positions of the discharge point. In fact, for slopes $\alpha \leq 60°$, it is entirely tolerable to replace the length s_o by the straight distance from $\overline{AB} = \sqrt{h^2+d^2}$, thus eliminating trial constructions. The construction is very similar to that shown in Figure 7b, except that point 1 is found by rotating distance $\overline{C_1B}$, or $\overline{AB}$, around point A.

If deviations up to 25 per cent. are permitted, the simplified value $s_o = \sqrt{h^2+d^2} = \overline{AB}$ may be used also for slopes up to 90°. For a vertical slope the formula for a is reduced to the following simple form:

$$a = \sqrt{h^2+d^2} - d \qquad (13)$$

In other words, for a vertical discharge face the height of the discharge point for the line of seepage can be approximated by the difference between the distance $AB = \sqrt{h^2+d^2}$ and its horizontal projection d.

c. Solution for a Horizontal Discharge Surface ($\alpha = 180°$). In 1931, Professor Kozeny (6) published a rigorous solution for the two-dimensional problem of ground water flow over a horizontal impervious surface which continues at a given point into a horizontal discharge face, as shown in Figure 9d. Kozeny's theoretical solution yields, for the flow lines and equi-potential lines, two families of confocal parabolas, with point A, where the impervious and pervious sections meet, as the focus.

The equation for the line of seepage can be conveniently expressed in the following form:

$$x = \frac{y^2 - y_o^2}{2y_o} \qquad (14)$$

in which x and y are the coördinates with the focus as origin, and y_o the ordinate at the focus $x = 0$.

If the line of seepage is determined by the coördinates d and h of one known point, then the focal distance a_o and the ordinate y_o are computed from the following equation:

$$a_o = \frac{y_o}{2} = \frac{1}{2}\left(\sqrt{d^2+h^2} - d\right) \qquad (15)$$

for which a graphical solution is recommended. (See Figure 11c.) The quantity y_o is simply equal to the difference between the distance $\sqrt{d^2+h^2}$ from the given point (d, h) to the focus of the parabola, and the abscissa d. The focal distance a_o is equal to one-half the ordinate y_o.

In addition to these simple relationships it is of advantage to remember that the tangent to the line of seepage at $x = 0$ and $y = y_o$ is inclined at 45°.

The quantity of seepage per unit of width is, according to Kozeny's solution:

$$q = 2ka_o = ky_o \qquad (16)$$

It is indeed fortunate that the problem of seepage with a horizontal discharge face has such a simple solution, not only because of the fact that in modern earth dam and levee design horizontal drainage blankets in the downstream section are assuming considerable importance, but also because this solution permits fairly reliable and simple estimates for the position of the line of seepage for overhanging discharge slopes.

d. Approximate Solutions for Overhanging Discharge Surfaces ($90° < \alpha < 180°$). Although the determination of the line of seepage and its point of exit for an overhanging discharge face, such as a rock fill toe, is of importance in the design of earth dams, little attention has been paid to this problem. Experimental results were published by Leo Casagrande (10 and 11), which permit a reasonably accurate determination of the line of seepage. Later, in 1933, the author checked the results of these model

tests by means of graphical solutions, of which a few typical examples are shown in Figure 9. These solutions check well with the experimental results just referred to, and are illustrated for one example in Figure 10. Such studies convinced the author that Forchheimer's graphical method for the determination of the flow net can be utilized for the solution of seepage problems with a free surface. The application of the graphical method to such problems requires considerable skill. This can be acquired only by extensive use of this method. Solutions such as those shown in Figure 9

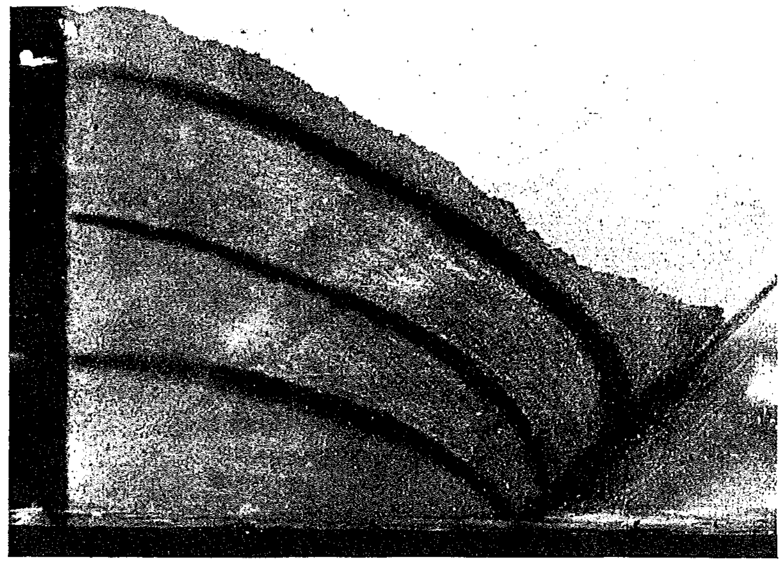

FIG. 10.— MODEL TEST ON OTTAWA STANDARD SAND WITH OVERHANGING DISCHARGE SLOPE.

Note zone of capillary saturation above line of seepage — upper dye line.
After L. Casagrande (10).

required many hours of work; sometimes several days were spent on one case,— because of the complications that the unknown upper boundary introduces into such seepage problems.

After sufficient graphical solutions to permit a rapid determination of the line of seepage for any slopes $60° \leq \alpha \leq 180°$, had been accumulated, the author's attention was called to Kozeny's (6) theoretical solution for $\alpha = 180°$. This proved a splendid opportunity for checking the accuracy of a purely graphical solution of a seepage problem with a free water surface. Figure 9d represents the original graphical solution. The difference between this solution and the theoretical solution is not more than 3 per cent. for any point on the line of seepage. Therefore, no attempt was made to include the theoretical solution in Figure 9d. This remarkable accuracy of the graphical method should convince critics that the method is not a plaything but has great merit and that the time spent on acquiring sufficient skill in this method is well invested.

To simplify the application of the graphical solution for very steep and overhanging discharge slopes such as are shown in Figures 9, a, b and c, these flow nets were compared with Kozeny's theoretical solution for a horizontal discharge face. For the sake of simplicity, the line of seepage for $\alpha = 180°$, which is represented by equations (14) and (15), and Figure 9d, will be referred to as the "basic parabola."

In Figure 9, the basic parabola is plotted into every case illustrated. The basic parabola and the actual line of seepage approach each other very quickly and for practical purposes may be assumed to be identical for points whose ordinates h are less than their horizontal distances from the discharge point C. By comparing the actual line of seepage for a given discharge slope with the basic parabola, we find that the intersection of this parabola with the discharge face is a distance Δa above the discharge point of the line of seepage. The ratio $C = \frac{a}{a + \Delta a}$ (see Figure 9) gradually decreases with increasing angle α. The ratio C is equal to 0.32 for $\alpha = 60°$; for a vertical surface ($\alpha = 90°$) it is 0.26; and for $\alpha = 180°$ the ratio C is, of course, equal to zero.

In order to utilize these relationships for determining the line of seepage and the discharge point for steep vertical and overhanging slopes, there has been plotted in Figure 11 the relationships between the ratio $\frac{a}{a + \Delta a}$ and the angle α. The quantity $a + \Delta a$ is found by intersecting the basic parabola with the discharge slope, an operation that can be performed either graphically or mathematically. In both cases one computes or constructs first $y_o = \sqrt{d^2 + h^2} - d$, for the known or estimated starting point of the line of seepage. The graphical determination of the intersection C_o is usually preferred, since the basic parabola is needed for the determination of the line of seepage. The construction of the parabola is best performed in the manner illustrated in Figure 8. For tangent $\overline{CT}$, either the tangent at the vertex of the parabola, or the tangent under 45° at $x = 0$ and $y = y_o$ can be used.

The points on the curve representing the relation between α and $c = \frac{a}{a + \Delta a}$, in Figure 11b, are derived from the graphical solutions. Note how close to a smooth curve these points lie. This is another demonstration of the degree of accuracy that can be obtained by means of the graphical method.

The quantity c is not only a function of the angle α but it also varies somewhat with the relative position of points B_1, or B_2, and C_o (Figure 11d). The maximum variations in c, for the limits that would normally be en-

countered in earth dams, are about ±5 per cent. The curve in Figure 11b was determined for a relatively short distance from the entrance to the discharge point of the line of seepage, in consideration of the importance of stratification in earth dams which is discussed in the next chapter.

Having plotted the basic parabola and determined the discharge point by means of the $c - \alpha$ relation, Figure 11, and knowing the tangent to the seepage line at the discharge point, it is an easy matter to draw with a fair degree of approximation the entire line of seepage, as shown in the various cases in Figure 9.

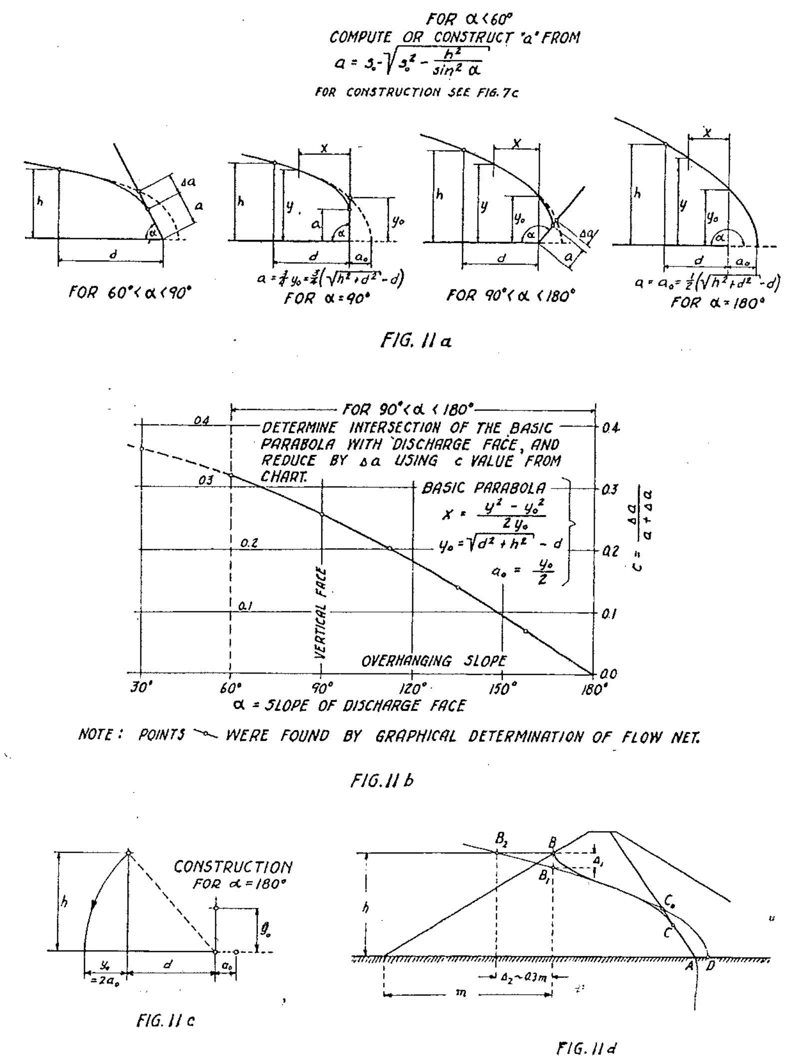

Fig. 11.—Application of Basic Parabola to Determination of Discharge Point of Line of Seepage.

e. Correction for Upstream Slope; Quantity of Seepage. Due to the entrance condition for the line of seepage and due to the fact that Dupuit's assumption is not valid for the upstream wedge of a dam, the line of seepage deviates from the parabolic shape. For the usual shape of a dam, there is an inflection point with a sharp curvature in the first section of the line of seepage, while for a vertical entrance face there is only an increase in curvature without reversal of direction.

For an accurate seepage analysis, these deviations should be taken into consideration. Referring to Figure 11d, it would be necessary to know in advance the position of one point of the parabolic curve in the vicinity of the entrance point B. L. Casagrande has chosen the intersection B_1 of the ordinate through the entrance point with the continuation of the parabolic line of seepage and has expressed the correction $\overline{BB_1} = \Delta_1$ as a function of d, h and the slope of the entrance face. A graphical presentation (11) facilitates the finding of the necessary correction Δ_1.

Somewhat simpler is the following approach. Instead of selecting point B_1 for the start of the theoretical line of seepage, we choose its intersection B_2 with the upstream water level. The corresponding correction Δ_2 is about $\frac{1}{3}$ to $\frac{1}{4}$ of the horizontal projection m of the upstream slope, or for average conditions $\Delta_2 = 0.3m$. This is easy to remember and dispenses with the necessity for tables or graphs for the correction. The determination of the line of seepage is then carried out with point B_2 as the starting point. The actual shape of the first portion of the line of seepage, starting at point B, can easily be sketched in, so that it approaches gradually the parabolic curve, as shown in Figure 11d.

The quantity of seepage q per unit of length can be computed either from equations (12) or (16). If we substitute in these equations the known quantities, they appear in the following form:

$$q = k(\sqrt{h^2+d^2} - \sqrt{d^2-h^2\ cotan^2\,\alpha})\ sin^2\,\alpha \qquad (17)$$

and

$$q = k(\sqrt{d^2+h^2} - d) \qquad (18)$$

For the great majority of cases encountered in earth dam design, both equations give practically the same result, so that the simpler equation (18) should be used for general purposes. In other words, the quantity of seepage is practically independent of the discharge slope, and is equal to the quantity that corresponds to the basic parabola. Only in those cases in which the starting point of the line of seepage is very near the discharge face will the difference between the two equations warrant the use of equation (17).

For the comparatively rare case in which the presence of tail water must be considered in the design, the determination of the line of seepage and of the quantity can be performed by dividing the dam horizontally at tail water level into an upper and lower section. The line of seepage is determined for the upper section in the same manner as if the dividing line were an impervious boundary. The seepage through the lower section is determined by means of Darcy's law, using the ratio of the difference in head over the average length of path of percolation as the hydraulic gradient. The total quantity of seepage is the sum of the quantities flowing through the upper section and the lower section. The results obtained by this rather crude approximation agree remarkably well with the values obtained from an accurate graphical solution.

Those readers who are interested in data showing how the results of seepage tests agree with the computed line of seepage and seepage quantity, using the methods described in this chapter, should consult References 10 and 11.

G. Seepage through Anisotropic Soils.

By a combination of the various methods of approach that have been outlined, and with proper consideration of boundary conditions as summarized in Figure 6 and discussed more in detail in Appendix I, one can arrive at a reliable determination of the line of seepage through even the most complicated cross-sections of earth dams. The cross-section may consist of portions with widely different permeabilities; however, each homogeneous section in itself is assumed to be isotropic, that is, possessing the same permeability in all directions. Unfortunately, this is practically never the case. Even a uniform clean sand, consisting of grains of the usual irregular shape, when placed in a glass flume for the purpose of building up a model dam section, does not produce an isotropic mass. The grains orientate themselves in such a manner that the coefficient of permeability is not uniform in all directions but larger in a more or less horizontal direction. As a consequence, the entire flow net is markedly influenced, resulting in considerable deviations from the theoretical flow net for an isotropic material.

Only by using a very uniform sand consisting of spherical grains, and by making tests on a sufficiently large scale, to reduce the capillary disturbance, can one arrive at test results that are in good agreement with theory. For this reason most of the tests described in References 10 and 11 were carried out on Ottawa standard sand.

Soils, in their natural, undisturbed condition, are always anisotropic in regard to permeability even if they convey to the eye the impression of being entirely uniform in character. If signs of stratification are visible, then the permeability in the direction of stratification may easily be ten times greater than that normal to stratification. For distinctly stratified soils this ratio can be very much larger than ten.

When soils are artificially deposited, as in the construction of a dam or dike, stratification develops to a greater or less degree. Such stratification has always been recognized by engineers as being undesirable, and for this reason special construction methods have been developed to disturb or destroy it. The hydraulic-fill core, during its construction, is frequently stirred with long rods in order to break up stratification as much as possible.

Sheepsfoot rollers are effective in compacting earth fills without creating distinct stratification. However, in spite of such precautionary measures, a certain amount of stratification remains. In addition, it is practically impossible to eliminate considerable variations in the general character of the material in the borrow pit; especially variations in permeability, which will result in substantial variations in the permeability of the dam from layer to layer. These cannot be eliminated by thorough rolling. Even the most carefully constructed rolled earth dams possess a considerably greater average permeability in a horizontal than in a vertical direction. Therefore, thorough investigation of variations in the character of the borrow pit materials forms an important part of preliminary studies. Taking into consideration the uncertainties that are always encountered in dealing with soil deposits and cannot be completely eliminated by the most elaborate investigations, it is essential that we should be conservative in the assumptions on which the design of an earth dam is based. This requires special attention to the possible degree of anisotropy in the dam.

The question of seepage through anisotropic soils was investigated for the first time and solved by Samsioe (14) in 1930. Fortunately the solution is simple and lends itself readily to practical application. The flow net of an anisotropic soil does not possess the usual characteristics of a flow net. However, it can be reduced, by the application of an appropriate geometric transformation, to an ordinary flow net. Designating the maximum and minimum coefficient of permeability for an anisotropic soil as k_{max} and k_{min}, it can be shown mathematically (see Reference 16) that by transforming the entire cross-section in such a manner that all dimensions in the direction of k_{max} are reduced by the factor $\sqrt{\frac{k_{min}}{k_{max}}}$, or that all dimensions in the direction of k_{min} are increased by the factor $\sqrt{\frac{k_{max}}{k_{min}}}$, the problem is again reduced to a solution of Laplace's equation. In other words, the flow net in the transformed section has the same characteristic flow lines and equipotential lines as previously discussed in this paper. Among others, Forchheimer's graphical method and all approximate methods suggested in this paper are applicable to the transformed section. After having found the line of seepage, or the entire flow net, in the transformed section, it is a simple matter to project this characteristic flow net back into the true section, in which flow lines and equipotential lines will not generally intersect at right angles. It should be noted that the hydraulic gradient at any point of the flow net and the magnitude of seepage pressures can only be determined in the true section, while the distribution of pore pressures and of hydrostatic uplift can be derived from either section.

The quantity of seepage can be computed from the transformed section on the basis of the coefficient of permeability $\bar{k} = \sqrt{k_{min} . k_{max}}$. For proof see Reference 16.

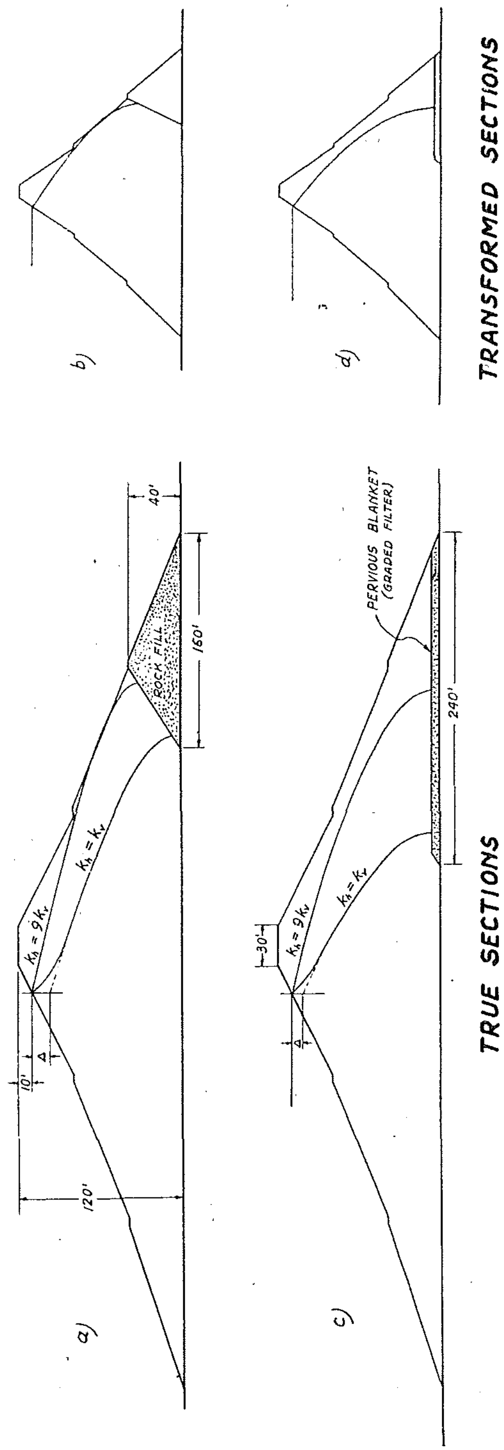

FIG. 12.—TRANSFORMATION METHOD FOR ANALYSIS OF STRATIFIED DAM SECTIONS.

Further information on seepage through stratified soils, the transformation theory, and examples may be found in References 15, 16 and 17.

The application of the transformation method is illustrated by the simple example of a rolled earth dam with a rock fill toe, shown in Figure 12. The dimensions and slopes of this dam are such that if a suitable soil is used, hardly any doubt would be raised regarding its stability. The rock fill toe seems to represent ample provision for safe discharge of seepage water. Indeed, the line of seepage, assuming isotropic soil, does fall well within the downstream face as shown in Figure 12a, (coefficient of permeability in horizontal direction k_h equal to coefficient in vertical direction k_v). However, if this dam is carelessly built of various types of soils with widely different permeabilities, a structure may well result that is many times more pervious in the horizontal direction than in the vertical direction. In the example shown in Figure 12, $k_h = 9k_v$ was chosen. On the right-hand side a new cross-section of the dam is plotted in which all horizontal dimensions are reduced by the factor $\sqrt{\frac{k_v}{k_h}} = \frac{1}{3}$. Then the line of seepage is determined in accordance with the methods outlined previously, and projected back into the true cross section. As can be seen in Figure 12a, the line of seepage for $k_h = 9k_v$ does intersect the downstream face, which is an undesirable condition that may in the course of time lead to a partial or complete failure of the structure.

H. Remarks on the Design of Earth Dams and Levees.

The question may arise of how to construct the downstream portion of a simple rolled earth dam, so that the line of seepage will remain a safe distance inside the downstream face of the structure, when only small quantities of coarse material are available. A simple solution is suggested in Figure 12c, in which a pervious blanket below the downstream portion of the dam is employed to control the position of the line of seepage to any desired extent. Such a blanket should be built up as a graded filter, carefully designed, to prevent erosion of any soil from the dam.

Whenever a dam or levee consists essentially of a uniform section of relatively impervious soil, e.g., possessing an average coefficient of permeability of less than 1 x 10^{-4} cm. per sec., the pervious blanket may well be extended as far as the centerline of the structure, as shown in Figure 13c. Such a design would add much more to the stability of the entire downstream portion, including the underlying foundation, than could be accomplished by a substantial flattening of the downstream slope. *A levee built in the conventional manner, with a downstream slope of 1 on 5, would possess less stability than a well-compacted levee in which the downstream slope is made as steep as 1 on 2, but which contains a filter blanket of the type shown in Figure 13c.* In the example illustrated in Figure 13c, it was assumed that a pervious foundation stratum lies beneath the levee and that the permea-

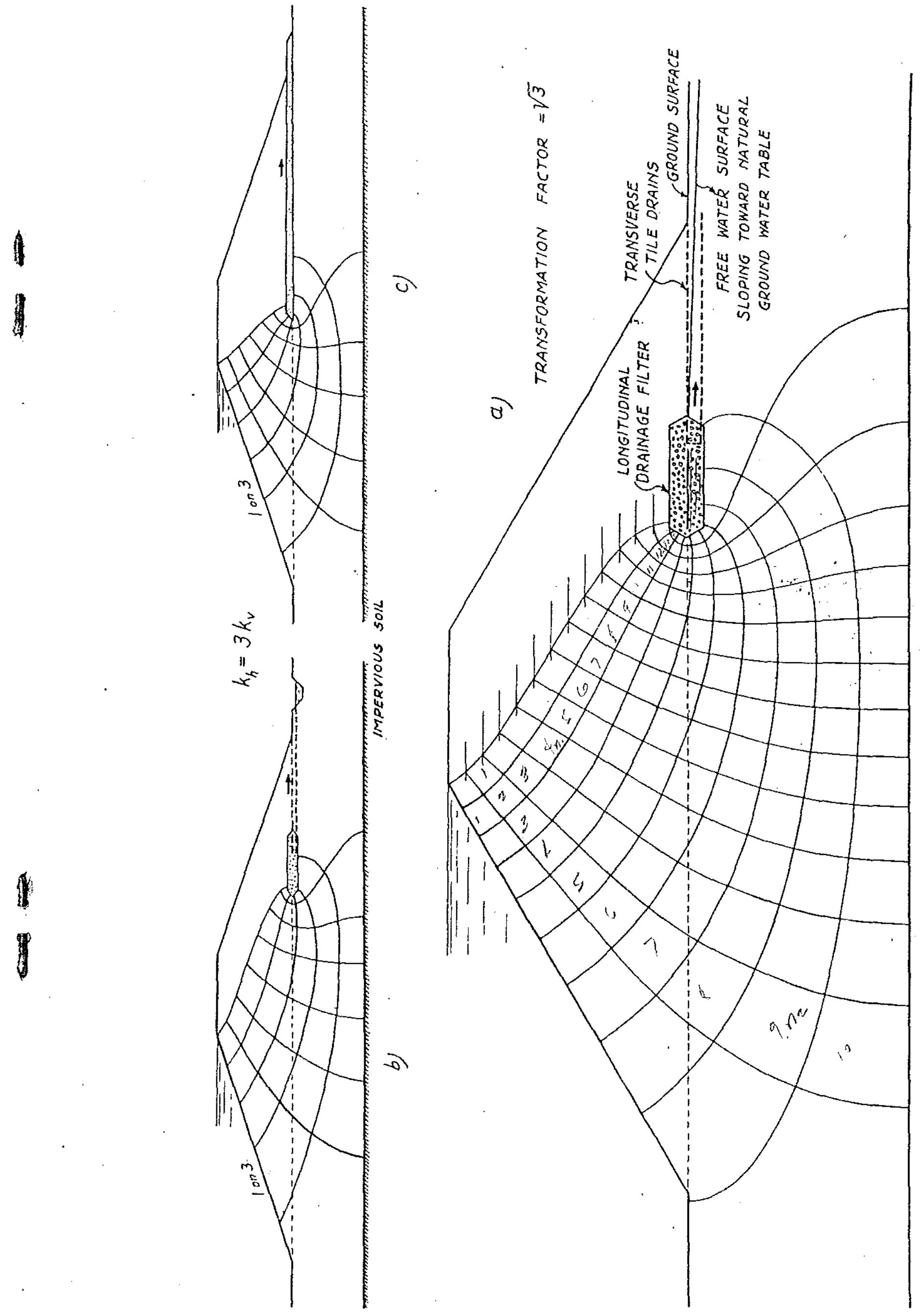

Fig. 13.— Effect of Internal Drainage on Flow Net.

bilities of the levee material and the underlying foundation are the same, with a ratio of $k_h/k_v = 3$. The transformation which was required for obtaining the flow net, is illustrated by another example shown in Figures 13a and b. The assumptions are the same as in Figure 13c, except that the filter blanket is reduced to a longitudinal drainage strip with frequent transverse outlets. A tile drain may also be embedded within the core of the longitudinal drainage filter to increase the capacity of the drainage system if the structure consists of relatively pervious soils. This type of drainage would be employed where the quantities of suitable material for the drainage layer are very limited. To obtain the flow net, the true section was transformed into a steeper section, using as transformation factor $\sqrt{k_h/k_v} = \sqrt{3}$. The flow net was then obtained by Forchheimer's graphical method, by gradual approximation. Note the equi-distant horizontal lines intersecting the line of seepage. These were plotted before starting the flow net. In this example the line of seepage is not identical with the basic parabola, because the surface of the foundation on which the structure rests is not a flow line. However, it is convenient to use the basic parabola as a general guide for the first plot of the flow net. After a satisfactory solution is found, the flow net is projected back into the true cross section, Figure 13b.

No attempt is made in this paper to discuss in detail the important and interesting relationships that exist between the stability of earth dams or dikes and the seepage through and beneath them. However, it should be emphasized that the forces exerted by percolating water upon the soil can be very appreciable, and are often a maximum in critical points. These seepage forces are readily determined from a well-constructed flow net, and can then be combined with gravity forces for the stability analysis. Anyone who has made comparative studies of the seepage forces that may exist in dams and their foundations, must be impressed by the paramount importance of the design of those features that control seepage. It is not surprising that on the basis of empirical knowledge levees have been constructed with flat slopes. A levee built in the conventional manner of sandy soil, with slopes of 1 on 2 or 1 on 3, would be an unsafe structure. However, substantial flattening of the slopes is a very costly way of increasing its safety. Besides, even very flat slopes do not necessarily provide sufficient safety against undermining, particularly when a levee rests on a stratified, pervious foundation. In view of the large expenditures on levee construction which the next decade will bring, investment in research in this field would pay rich dividends if new designs for levees were developed that would not only be much safer than those built in the past, but considerably less expensive. *The widespread opinion among engineers that in earth dam and levee design "section makes for safety" needs to be revised.*

Many failures of levees are due to undermining caused by seepage through the foundation. Unless drainage provisions, as shown for example in F[illegible] 13, are provided, the largest concentration of flow lines, both

through and beneath the dam, occurs at the downstream toe where the soil is not confined. As stated before, flattening the slopes does not greatly improve this condition. In some cases, particularly when the foundation soils are porous and distinctly stratified, drainage provisions within a levee may not be sufficient protection. In such circumstances it may be beneficial to drill frequent holes into the foundation beneath the future longitudinal drain, and to fill these holes with coarse material. Such "drainage wells" have been employed already for another purpose, the relief of upward pressure on an overflow dam (see Reference 18). Properly designed and constructed drainage wells would effectively destroy any serious flow concentration at the toe of the structure. In other cases, particularly for a very pervious, but relatively thin foundation stratum, the use of a sheet-pile cut-off may be the ideal solution.

Improvements in levee design, as suggested here, are of a nature that will not produce interference with modern construction methods. Drainage wells, longitudinal and transverse drains or drainage blankets must all be built before construction of a levee is started. The building of the levee can then proceed in the same manner as if the drainage structures did not exist, thus permitting full use of large drag lines and tower machines.

I. Seepage through Composite Sections.

For the purpose of controlling seepage and utilizing available soils to best advantage, it is usually necessary to build dams of several sections with widely different coefficients of permeability. Since it is common to require that the ratio in permeability between neighboring sections should be at least one to ten, it is rarely necessary to determine the flow net for the entire dam if a careful study is made of the least pervious sections. However, in some cases it may be necessary to determine the position of the entire line of seepage. In Figure 14 is reproduced an example given by L. Casagrande (12), showing the line of seepage for a combination of two sections, with the downstream section built of soil which is five times more

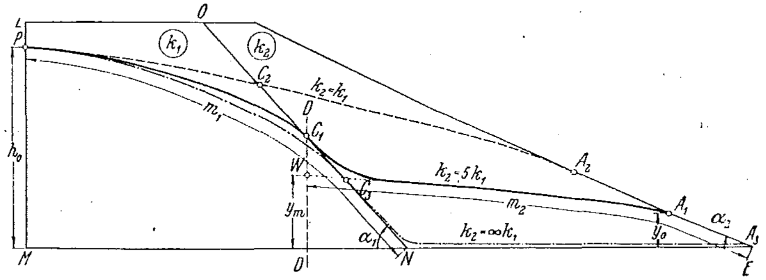

FIG. 14.—LINE OF SEEPAGE FOR A COMBINED SECTION.
The computed result was verified by model experiments.
After L. Casagrande (11).

pervious than the central section. The line of seepage through such a composite section is found by changing the assumed position for the point of intersection of the line of seepage with the boundary until the quantities flowing through both sections are the same. After the correct position of the line of seepage has been determined, one can also develop any desired portions of the flow net.

J. Comparison between Forchheimer's Graphical Method, Hydraulic Model Tests and the Electric Analogy Method.

The purpose of model tests for seepage studies can be twofold (1) determination of the flow net for a given cross-section, assuming that the soil is isotropic; (2) determination of the flow net if the model is build up in such a manner that it resembles the prototype as to possible stratification, character of the soils, etc.

For the first purpose it is essential that the material used shall consist of grains as nearly spherical and as nearly of one size as possible. In addition, the models must be large enough so that the height of capillary rise will not distort the line of seepage, particularly in that portion of the flow net near the discharge point of the line of seepage. (See Figure 10.) Use of Ottawa Standard Sand has given good results. Since one should not go much below the size of Ottawa Standard Sand on account of the distorting effect of capillary rise, one is obliged to use artificial spheres, such as glass spheres, of appropriate sizes for models containing sections with different coefficients of permeability. For such coarse materials the validity of Darcy's law must be checked. The results of careful model tests conducted with such materials agree well with the solutions obtained by the graphical method, the electric analogy method, or rigorous theoretical solutions, so far as the latter are available.

For the second purpose, the testing of models similar to the prototype, one has to know first of all how the coefficient of permeability varies in the prototype, not only in its various sections, but particularly within each section due to anisotropy. Then one must build the model to imitate, on a small scale, these conditions. It is a waste of time and money to build a model using the same soil as in the prototype without attention to the anisotropic conditions in the prototype. Such a model does not represent the prototype; nor are the results comparable to the conditions for isotropic materials, because the inevitable irregularities and stratification due to the method of building the model are reflected in the resulting flow net to such an extent that the flow net looks very much like a beginner's attempt at employing the graphical method. The results of such model tests will lie somewhere between the conditions for an isotropic model and the actual conditions in the prototype and will tell practically nothing that can be of assistance in our problem; on the contrary, such

results are often very confusing, particularly when the tests are made by men inexperienced in theoretical and graphical solutions and, therefore, are unable to interpret properly the test results.

The electric analogy method, when used by an experienced operator, is a useful and accurate method for *direct determination of flow nets between given boundaries.* Composite sections consisting of soils with different permeability can also be investigated by this method. Unfortunately, the method does not permit the *direct* determination of the line of seepage for those problems in which the upper surface is not a fixed boundary. Another disadvantage of this method is that it requires an accurate apparatus and the construction of a special testing model for every problem. If compared with graphical solutions, the electrical method is more expensive and requires more time. It should also be mentioned that a satisfactory presentation of the results obtained by means of the electrical method requires knowledge and application of the graphical method. In some instances the amount of work required to transform the test results into a good-looking flow net would have been enough to produce this flow net by the graphical method without assistance of the electrical apparatus. The graphical method will serve as an excellent check on the electrical method and should be used whenever accurate solutions are sought.

Another important advantage of the graphical method is that the process of finding the flow net for a proposed section almost inevitably suggests changes in the design which would improve the stability of the structure, and often its economy. With some experience in the use of the graphical method the effects of changes in one or the other detail of the design can quickly be appraised without the necessity of finding the complete flow net for a number of different cross-sections. Thus there can be explored in a short time many possibilities which would require months of work with any of the other methods. Such studies have already indicated desirable changes from the conventional design of earth dams and levees, some of which were briefly discussed in the preceding chapter.

Finally, there should be mentioned the pedagogical value of the graphical method. It gradually develops a feeling or instinct for streamline flow which not only improves, in turn, the speed and accuracy with which flow nets can be determined, but also develops a much better understanding of the hydromechanics of seepage and ground water movement. The investigator who is trained only in the use of "mechanical" methods for analysing seepage problems can check his tests only by performing additional tests. He is rarely able to detect inaccuracies by the appearance of the test results. In contrast to this, the author has been able to point out even minor inaccuracies in the results obtained from model tests, as a result of the sense for streamline flow developed by applying the graphical method for years.

In concluding this discussion, the author wishes to emphasize the almost obvious point, which nevertheless is frequently overlooked, that the investigator should consider carefully, before starting any model tests, what information he desires to obtain from these tests. In nine cases out of ten he will then come to the conclusion that he could obtain the results without tests. Particularly in those cases where he attempts to evaluate the effect of variations in the coefficient of permeability, he will arrive at a better conception of the probable limits within which the seepage conditions in the prototype may vary by making a careful study of the possible variations in the coefficient of permeability (e.g. from studies of the variations in the borrow pit material) and then applying these values in graphical solutions, utilizing the transformation method. A model test would yield only one result, the relation of which to the prototype is often unknown. Such a test would certainly not permit a conclusion in regard to the probable limits within which the actual flow conditions will vary.

The practical application of the graphical method would be promoted if, for all typical conditions encountered in dam design, carefully constructed flow nets were published. The beginner in the use of the graphical method in particular, would be greatly assisted and encouraged in his efforts to acquire skill in the use of this valuable tool.

Appendix I.

(a) *Deflection of Flow Lines Due to Change in Permeability.* Flow lines are deflected at the boundary between isotropic soils of different permeability in such a manner that the quantity Δq flowing between two neighbouring flow lines is the same on both sides of the boundary. Referring to Figure 5, in which the flow net is plotted on the basis of squares for the material on the left of the boundary, and designating by Δh the drop in head between any two neighbouring equipotential lines, the following relationship can be set up:

$$\Delta q = k_1 a \frac{\Delta h}{a} = k_2 c \frac{\Delta h}{b}$$

or

$$\frac{k_1}{k_2} = \frac{c}{b} \qquad (19)$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \text{ and } \frac{a}{\cos\alpha} = \frac{b}{\cos\beta}.$$

By combining these relationships one arrives at:

$$\frac{c}{b} = \frac{\tan\beta}{\tan\alpha} = \frac{k_1}{k_2} \qquad (20)$$

Expressed in words, the deflection of the flow lines occurs such that the tangent of the intersecting angles with the boundary is inversely proportional to the coefficients of permeability. Furthermore, the squares on one side of the boundary change on the other side into rectangles with the ratio of their sides equal to the ratio of the coefficients of permeability, such

that the flow channels are wider in the material with the smaller coefficient of permeability.

It is probable that Forchheimer was the first one to use these relationships. However, he never took the trouble to publish them. In 1917, he communicated those relationships to Terzaghi, who made extensive use of them in his foundation investigations of dams, and also taught them in his course in Soil Mechanics at the Massachusetts Institute of Technology, during 1925–29.

(b) *Transfer Conditions for Line of Seepage at Boundaries; General Remarks.* L. Casagrande (10) made use of the general properties of a flow

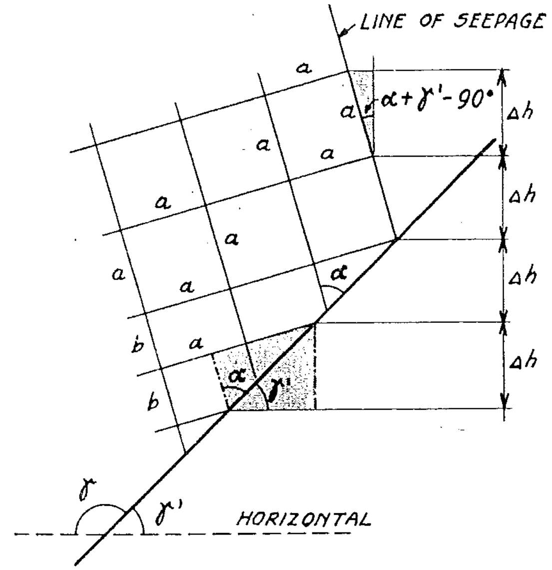

Fig. 15.— Derivation of Discharge Condition into Overhanging Slope.

net to analyze the condition at the entrance and discharge points of the line of seepage. Following the same general approach, the author determined the transfer conditions for other cases, including the transfer at the boundary between soils of different permeability. The results are assembled in Figure 6.

To acquaint the reader with the method used, it will be sufficient to present in the following the derivation for two typical cases.

(c) *Discharge into an Overhanging Slope.* In Figure 15 is shown the flow net in the immediate vicinity of the discharge point, sufficiently enlarged so that flow lines and equipotential lines appear straight. The slope of the line of seepage at the discharge point, the "discharge gradient," is assumed arbitrarily; then the flow net is plotted, starting with a series of equidistant horizontal lines which represent the head of consecutive equipotential lines. One can see immediately that the assumed discharge gradient in Figure 15 cannot be correct, because it is impossible to draw squares in the lower portion of the flow net. By setting up the condition that the sides a and b of the resulting rectangles must become equal, one can arrive at the necessary condition for the discharge gradient.

By projecting the sides a and b in the shaded triangles, Figure 15, one arrives at the following equation:

$$\frac{b}{\cos\alpha}\sin\gamma = a\cos(\alpha+\gamma-90°) = \Delta h$$

To fulfill the condition $a = b$, the only possible solution is $\alpha = 90-\gamma$; that means the line of seepage must have a vertical discharge slope.

(*d*) *Transfer Conditions for Line of Seepage at Boundary between Soils of Different Permeability.* To analyze the transfer conditions for the cases illustrated in Figures 6k and m, we start from the conditions that the hydraulic gradient at any point along the line of seepage is equal to the sine of the slope of the line of seepage of that point; and that the quantity flowing through a very thin flow channel along the line of seepage must be equal on both sides of the boundary. Referring to Figure 16a, we have the following velocities along the line of seepage, on both sides but in the immediate vicinity of the boundary:

$$v_1 = k_1 \sin(\alpha-\omega')$$
$$v_2 = k_2 \sin(\beta-\omega')$$

The quantity Δq flowing through the channel is:

$$\Delta q = a k_1 \sin(\alpha-\omega') = c k_2 \sin(\beta-\omega')$$

wherein the quantities a and c represent the widths of the flow channels, in accordance with Figure 16a. After replacing the quantities a and c by their projection onto the boundary, and substituting $k_1/k_2 = \tan\beta/\tan\alpha$, one arrives at the general condition:

$$\frac{\cos\alpha}{\cos\beta} = \frac{\sin(\beta-\omega')}{\sin(\alpha-\omega')}$$

or

$$\frac{\sin(90-\alpha)}{\sin(90-\beta)} = \frac{\sin(\beta-\omega')}{\sin(\alpha-\omega')}$$

If we assume $90-\alpha > \beta-\omega$, then it follows that $90-\beta < \alpha-\omega'$, and vice versa. Hence the only possi ble solution is:

$$\left.\begin{aligned} &90-\alpha = \beta-\omega' \\ \text{or}\quad &\beta = 90+\omega'-\alpha \\ \text{or}\quad &\beta = 270°-\alpha-\omega \end{aligned}\right\} \qquad (21)$$

Since this condition does not contain the coefficients of permeability, it has to be fulfilled simultaneously with equation (20). Equations (20) and (21) determine, for a given slope ω of the boundary, the unknown angles α and β between the line of seepage and the boundary.

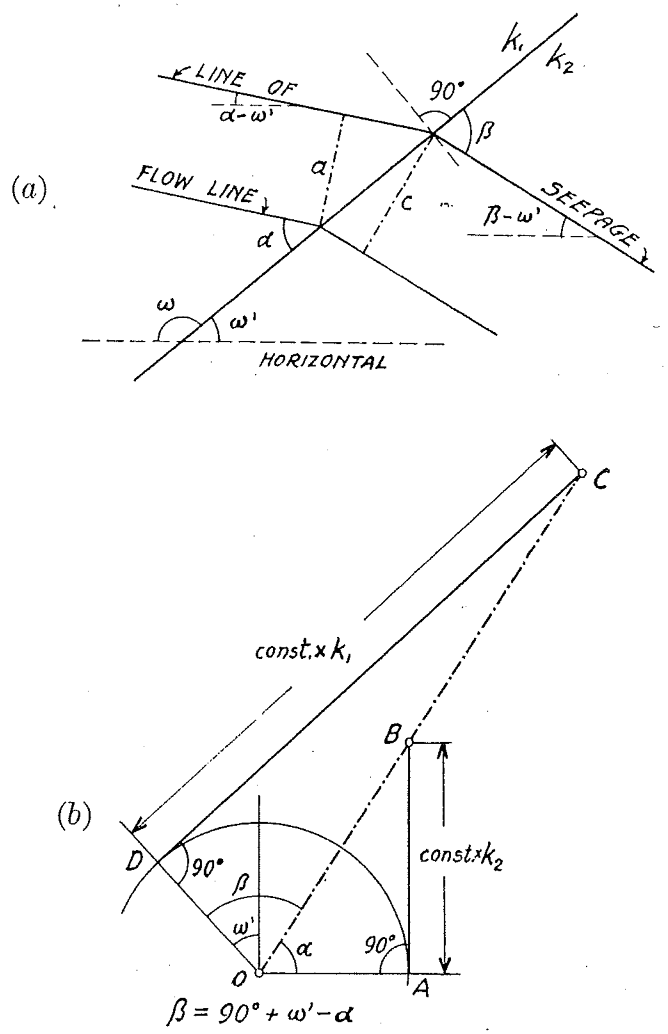

FIG. 16.—TRANSFER CONDITION OF LINE OF SEEPAGE AT OVERHANGING BOUNDARY.

The solution of these two equations can best be found graphically in the manner illustrated in Figure 16b. A circle is drawn with an arbitrary radius and the angle AOD is made equal to $(90+\omega')$. Then lines are drawn through points A and D perpendicular to the corresponding radii. The problem is to draw another line through the center O (shown as dot-dash line) which fulfills the condition that the ratio $\overline{AB}/\overline{CD} = k_2/k_1$. Such a line can be found quickly by trial. The unknown angles α and β are determined by the angles between the dot-dash line and lines OA and OD, respectively.

Depending on whether k_1 is larger or smaller than k_2, we arrive at solutions in which either point B or point C is nearer the center of the circle. The corresponding deflection of the line of seepage is illustrated in Figures 6k and m.

While this theoretical solution for $k_1 > k_2$ can easily be verified by model experiments, it is not generally true for $k_1 < k_2$. In this case, when the downstream section is more pervious, the boundary condition for the line of seepage is also influenced by all other dimensions of the dam, especially by the elevation of the discharge point and its distance from the boundary under consideration. Only in special cases, particularly for high tail water

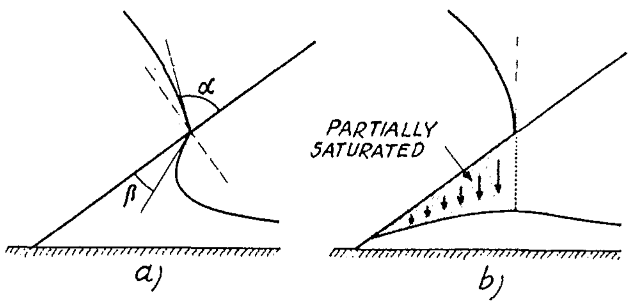

FIG. 17.—TRANSFER CONDITION OF LINE OF SEEPAGE AT OVERHANGING BOUNDARY.

level, and coefficients of permeability that do not differ greatly, does the line of seepage follow the theoretical solution. Whenever the theoretical solution has the appearance shown in Figure 17a, with the line of seepage deflected into an overhanging slope, it represents a condition that may be observed on a small scale in the laboratory but does not occur on a large scale. Instead of the continuous line of seepage of Figure 17a, a discontinuity develops, with the water seeping vertically into the more pervious soil, and only incompletely filling its voids. In other words, the quantity discharging vertically downward at the boundary is insufficient to fill the voids of the coarse material. Therefore, normal atmospheric pressure will act along that section of the boundary and the laws for open discharge are valid, forcing the line of seepage to assume a vertical discharge gradient at the boundary. That portion of the coarser soil which is only partially saturated, is illustrated in Figure 17b by the shaded area.

The graphical solution shown in Figure 16b also permits determination of the transfer conditions for the entrance of the line of seepage for the special case illustrated in Figure 6c. The open body of water on the up-

stream side may be considered a porous material with $k_1 \rightarrow \infty$, for which case Figure 16b yields $\beta = 90°$. The same conclusion may be reached from equation (21), remembering that for this case, $\alpha = \omega'$. This result means that the line of seepage enters perpendicularly to the upstream face of the dam, as shown in Figure 6c.

(*e*) *Singular Points in a Flow Net.* In trying to apply Forchheimer's graphical method, the beginner is frequently puzzled by the fact that some

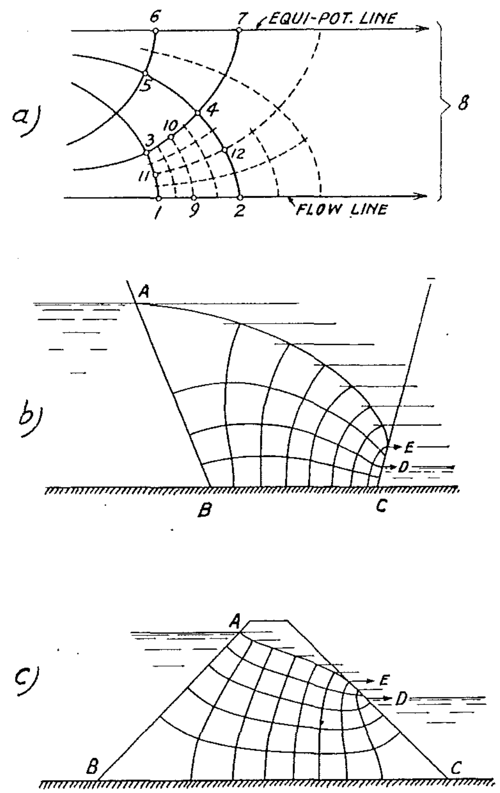

FIG. 18.—ILLUSTRATIONS OF SINGULAR POINTS OF FLOW NETS.

"squares" have no resemblance to real squares, and that, in some cases, flow lines and equipotential lines do not intersect at right angles. For example, in Figure 18a, the full-drawn areas 1, 2, 3, 4 and 4, 5, 6, 7 do not appear like "squares" to the inexperienced. However, by subdividing such areas by equal numbers of auxiliary flow lines and equipotential lines, one can easily check whether the original area is a "square" as defined for flow nets. By such sub-division one must arrive at areas which appear more and more like real squares. However, in most cases it is sufficient to compare the average distances between opposite sides, that is, e.g., the lengths 9–10 and 11–12, by means of a pair of dividers.

In Figure 18a, the entire area to the right of points 2, 4, 7 must also be considered a square, in spite of the facts that the fourth point lies at infinity and that the angle between the flow line and equipotential line at this point is zero instead of 90°. It is, indeed, possible to continue subdividing this area, as shown by the dotted lines, always leaving a semi-infinite strip as the "last square." By this process of subdivision the amount of water entering into the "last square" is continuously reduced and approaches zero. In this way it is possible to reconcile the irregularity of the fourth corner by the fact that there is no flow of water at that point.

Similar irregularities in the shape of squares appear wherever a given boundary of the soil, with water entering or discharging, and boundary flow lines (impervious base or line of seepage) intersect at a predetermined angle. If this angle is less than 90°, then the velocity of the water at the point of intersection is zero. Such points are the entrance point A of the line of seepage in Figure 18b, and points B and C in Figure 18c. On the other hand, if the intersecting angle is greater than 90°, then the theoretical velocity in that point is infinite. Such points are corner A in Figure 9d, corners B and C in Figure 18b, and point D in Figure 18c. The last, representing the concentration of flow lines at the elevation of tail water level, is the cause for the well-known erosion which is observed on the downstream slope of homogeneous dam sections at the line of wetting.

At points where the theoretical velocity is infinite, the actual velocity is influenced by the facts that for larger velocities Darcy's law loses its validity, and that changes in velocity head become so important that they cannot be neglected. Hence, in the vicinity of such points the general differential equation (4) is not valid, and the flow net will deviate from the theoretical shape. However, the areas affected are so small that these deviations may be disregarded.

Appendix II.

Additions to the Original Paper.

(*a*) *Graphical Procedure for Determining Intersection between Discharge Slope and Basic Parabola.* The intersection between the discharge face and the basic parabola, designated in Figure 9 by point C_o, can be determined by the following simple graphical procedure.

The ordinate h_1 of the intersection of the basic parabola $x = \frac{y^2 - y_o^2}{2y_o}$ with the discharge slope $y = \pm x \tan \alpha$ is found as the solution of these two equations in the following form:

$$h_1 = \pm \frac{y_o}{\tan \alpha} + \sqrt{\frac{y_o^2}{\tan^2 \alpha} + y_o^2}$$

The first member, $\frac{y_o}{\tan \alpha}$, is equal to the distance $\overline{EB} = f$, in Figure 19; the second member, under the square root, is equal to the distance $\overline{AB} = g$; the ordinate h_1 of the intersection is simply equal to the sum $(f+g)$ for angles $\alpha < 90°$, and equal to the difference $(g-f)$ for angles $\alpha > 90°$. These relationships are expressed by the construction shown in Figure 19, which needs no further explanation.

The discharge point of the line of seepage is then found as discussed in Section F-d, with the help of Figure 11.

(*b*) *Comparison between Hamel's Theoretical Solution and the Proposed Approximate Methods.* Hamel (19), has succeeded in arriving at a rigorous mathematical solution of the problem of seepage through a homogeneous

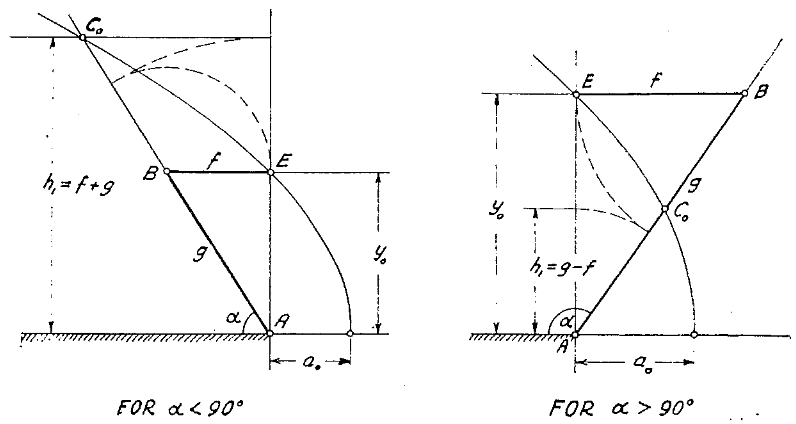

FIG. 19.—GRAPHICAL METHOD FOR DETERMINING INTERSECTION BETWEEN BASIC PARABOLA AND DISCHARGE FACE.

dam section. Unfortunately, the theory is so cumbersome that, in its present form, it is of little use to the engineering profession. It will be necessary to compute a sufficient number of typical cases, and publish the results in the form of tables or graphs, before engineers will be able to realize the advantages of this theoretical treatment. Recently, a few cases have been computed by Muskat (20) for coffer dam sections with vertical sides. These solutions presented an opportunity to investigate, at least for a few special cases, the accuracy of the approximate methods proposed in this paper. The results of this comparison were so encouraging that they are presented in the following paragraphs to permit the reader to formulate his own conclusions.

In Figure 20 are assembled three of the six cases which were published by Muskat (20). In each case the elevation of the discharge point, as computed from Hamel's theory, is designated by C_o, and its vertical distance from the tail water level, or from the impervious base in the absence of tail water, is designated a_H.

An approximate elevation of the discharge point was found by means of the graphical procedure shown in Figure 7c. To facilitate the comparison between these figures, all points in Figure 20 are marked to correspond to those in Figure 7c. The construction is shown with full lines. The resultant discharge point is marked C, and its elevation from the base, or the tail water level, is marked a.

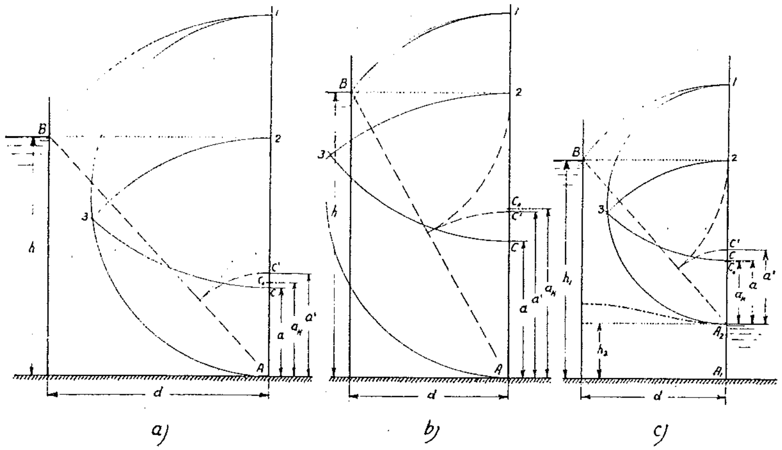

FIG. 20.— COMPARISON BETWEEN RIGOROUS AND APPROXIMATE DETERMINATIONS OF DISCHARGE POINT.

In addition to this construction, the simplified procedure was used in which $s_o = \sqrt{h^2+d^2}$. For a vertical discharge face the simplified formula for a becomes $a' = \sqrt{h^2+d^2}-d$ as proposed by Kozeny (6). The corresponding construction is shown in Figure 20 by dash lines and the resulting discharge point and elevation are designated by C' and a' respectively.

The case illustrated in Figure 20a, corresponding to Muskat's case No. 6, is identified by the ratio $d/h = 0.937$. Hamel's theory yields the quantities $a_H/h = 0.394$, and for the rate of seepage, $q_H = 0.539\ kh$.

As was shown by Muskat (20) and Dachler (23), the rate of seepage computed by means of Dupuit's formula $q = \frac{h_1^2 - h_2^2}{2d} k$, or without tail water $q = \frac{h^2}{2d} k$, represents an excellent approximation. For the case illustrated in Figure 20a, we have

$$q = \frac{h}{2d}\ kh = 0.539\ kh$$

The next case, Figure 20b, corresponds to Muskat's case No. 4 and is identified by the ratio $d/h = 0.556$. The theory by Hamel yields the quantities $a_H/h = 0.596$, and $q_H = 0.898\ kh$. The approximate rate of seepage, computed from Dupuit's formula, as described for the previous case, is $q = 0.900\ kh$.

The third case, shown in Figure 20c, is identical with Muskat's case No. 2. It differs from the other examples by the assumption of a definite tail water level, and is identified by the quantities $d/h_1 = 0.663$, and $d/h_2 = 2.81$. From the theory we get $a_H/h_1 = 0.301$, and $q_H = 0.717\ kh_1$. Dupuit's formula yields $q = 0.695\ kh_1$.

The comparison between the values for the elevation of the discharge point obtained by Hamel's rigorous solution and those by the approximate methods shows that, for engineering purposes, the approximate solutions are very satisfactory. It is interesting and of practical value to note that the approximate methods also give satisfactory results for ratios d/h considerably smaller than 1.0. Considering that the upstream portion of the flow net differs considerably from Dupuit's assumption of a constant hydraulic gradient in all verticals, this result is somewhat unexpected. For ratios of $d/h < 1.0$, it appears that Kozeny's formula (13) gives slightly better results than the formula by L. Casagrande.

The remarkable agreement between the theoretical rate of seepage and Dupuit's approximate solution deserves special emphasis.

(c) *Graphical Solution by Means of the Hodograph.* A new graphical method for determining the flow net was proposed by Weinig and Shields (30), in which the flow-net is determined graphically in the "hodograph plane" and then projected into the actual cross section.

The hodograph of a flow line is the curve which one obtains when plotting from one origin velocity vectors for all the points of the flow line. Therefore, the straight line connecting the origin with one point on the hodograph represents the magnitude and direction of the velocity for the corresponding point on the flow line.

Since the velocity along the free water surface is proportional to the sine of the slope, the hodograph for the line of seepage is a circle with diameter equal to the coefficient of permeability. The hodograph for a straight boundary is a straight line. Therefore, all boundaries of the hodograph that correspond to the flow net of a homogeneous isotropic dam section are known, and it is possible to set up equations that represent the solution of the problem in implicit form. That such a theoretical solution is rather complicated, even for the simplest dam section, has been mentioned before in the discussion of Hamel's theory (19). Therefore, Weinig and Shields follow the theoretical approach, using the hodograph, as far as mathematics permits conveniently; then they proceed to find the flow lines and equipotential lines in the hodograph by a graphical procedure which is essentially similar to Forchheimer's method.

One advantage of the method by Weinig and Shields is the possibility of determining numerically correct values for the velocity at certain points along the boundaries. In comparison with Forchheimer's graphical method the approach by Weinig and Shields is much more complicated and requires a thorough acquaintance with the hodograph, which very few engineers possess. Furthermore, this method is limited to simple cross-sections, while Forchheimer's method can be applied to complicated dam sections and foundation conditions.

Weinig and Shields (30) have solved a steep triangular dam section by means of the graphical solution of the hodograph. This cross-section is

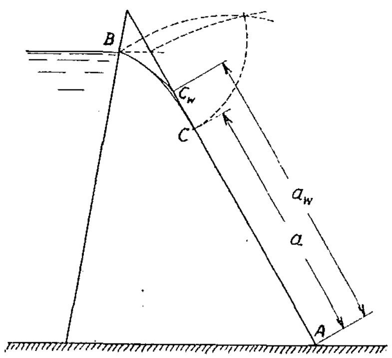

Fig. 21.— Discharge Points Obtained by Graphical Solution of Hodograph and Method Illustrated in Fig. 7.

illustrated in Figure 21. Point C_w represents the discharge point as determined from the hodograph, and point C the discharge point using the method shown in Figure 7b. The elevation of point C is 15 per cent. lower than that of C_w. How much of this difference is due to inaccuracy in one or the other method is uncertain. Probably the hodograph solution is more accurate when the entrance point of the line of seepage is very close to the discharge face. In Figure 20, as well as in Figure 21, the discharge points obtained by the simple graphical procedure are situated lower than the other more accurate solutions. This would indicate the necessity for applying a correction in those cases where upstream and downstream face are very close.

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